

MAS212 Linear Algebra I Exercises 10 (with solutions)

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Assessed submission deadline 5pm on Friday 15th December 2006 in the blue box on the ground floor.

Your submission must have your FULL NAME and STUDENT NUMBER PRINTED CLEARLY at the top and separate sheets must be stapled together.

Marks out of 100 are shown in square brackets after each question number.

The field of scalars is the complex field $\mathbb{C}$ in all cases.

1. [25] For the matrix $A = \begin{bmatrix} 4 & 1 & -2 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$:

- (a) construct the characteristic equation of A (by evaluating the determinant in a way that gives the characteristic polynomial in a factorised form if possible);
- (b) solve this equation to find the eigenvalues of A ;
- (c) for each eigenvalue, determine a corresponding eigenvector of A ;
- (d) write down an invertible matrix P^{-1} such that $P A P^{-1} = \Lambda$ is diagonal, write down what Λ should be, and then check that $P A P^{-1} = \Lambda$;
- (e) use Λ (or equivalently the eigenvalues) to compute the trace and determinant of A .

▼ Solution

(a) The characteristic polynomial of A is $\begin{vmatrix} 4-\lambda & 1 & -2 \\ -2 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix}$. Laplace expansion about the

bottom row give this in factorised form as $(1-\lambda)((4-\lambda)(1-\lambda) + 2)$. Expanding and factorising the second factor leads to the characteristic equation in the form

$$(1-\lambda)(\lambda^2 - 5\lambda + 6) = (1-\lambda)(\lambda-2)(\lambda-3) = 0.$$

(b) Hence the eigenvalues of A are $\lambda = 1, 2, 3$.

(c) Solving $(A - \lambda I)x = 0$ for x with $\lambda = 1, 2, 3$ gives the eigenvectors

$(1, -1, 1)$, $(-1, 2, 0)$, $(-1, 1, 0)$ (up to scalar factors).

(d) Using these eigenvectors as the columns gives $P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, which reduces A to the

diagonal matrix of eigenvalues $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Check:

$$P A P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

⌊ (e) $\text{tr}(A) = 1 + 2 + 3 = 6$ and $\det(A) = 1 \times 2 \times 3 = 6$.

► **Maple Solution**

2. [25] Repeat Question 1 with $A = \begin{bmatrix} 1 & 5 \\ -1 & 5 \end{bmatrix}$.

▼ **Solution**

(a) The characteristic equation of A is

$$\lambda^2 - 6\lambda + 10 = 0$$

(b) Solving using the quadratic formula, the eigenvalues of A are $\lambda = 3 \pm i$.

(c) Substituting each eigenvalue into the equation $(A - \lambda I)x = 0$ and solving the first scalar equation for x gives the eigenvectors $(5, 2 \pm i)$.

(d) Using these eigenvectors as the columns gives $P^{-1} = \begin{bmatrix} 5 & 5 \\ 2+i & 2-i \end{bmatrix}$, which reduces A to the

diagonal matrix of eigenvalues $\Lambda = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix}$. Check:

$$P A P^{-1} = \begin{bmatrix} \frac{1}{10} + \frac{1}{5}i & -\frac{1}{2}i \\ \frac{1}{10} - \frac{1}{5}i & \frac{1}{2}i \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 & 5 \\ 2+i & 2-i \end{bmatrix} = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix}.$$

(e) $\text{tr}(A) = (3+i) + (3-i) = 6$ and $\det(A) = (3+i)(3-i) = 9+1 = 10$.

But note that eigenvectors are not unique, and a possible alternative pair of eigenvectors that arise naturally from solving the second rather than the first scalar equation is $(2 \mp i, 1)$ giving

$P^{-1} = \begin{bmatrix} 2-i & 2+i \\ 1 & 1 \end{bmatrix}$. Check:

$$P A P^{-1} = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}-i \\ -\frac{1}{2}i & \frac{1}{2}+i \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2-i & 2+i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix}.$$

Note that $(5, 2 \pm i)$ is a (complex) scalar multiple of $(2 \mp i, 1)$ since $(2 \pm i)(2 \mp i, 1) = (5, 2 \pm i)$.

► **Maple Solution**

3. [25] Repeat Question 1 with $A = \begin{bmatrix} 3 & -2 & 3 \\ 1 & 0 & 3 \\ 1 & -2 & 5 \end{bmatrix}$. **Hints:** To solve the characteristic equation first

show that one of the eigenvalues is 4. It is possible to diagonalise a matrix that has non-distinct eigenvalues if enough eigenvectors can be *chosen* to be linearly independent.

Solution

(a) The characteristic equation of A is

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

which factorizes as

$$(\lambda - 4)(\lambda - 2)^2 = 0$$

(b) Hence the eigenvalues of A are $\lambda = 4, 2, 2$.

(c) Solving $(A - \lambda I)x = 0$ for x with $\lambda = 4$ gives the eigenvector $(1, 1, 1)$ and with $\lambda = 2$ gives the general solution $(2y - 3z, y, z)$, from which we can choose $(2, 1, 0)$, $(-3, 0, 1)$ as a basis for the 2-dimensional eigenspace.

(d) Using these eigenvectors as the columns gives $P^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, which reduces A to the

diagonal matrix of eigenvalues $\Lambda = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Check:

$$P A P^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & \frac{3}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 3 \\ 1 & 0 & 3 \\ 1 & -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(e) $\text{tr}(A) = 4 + 2 + 2 = 8$ and $\det(A) = 4 \times 2 \times 2 = 16$.

Maple Solution

4. [25] The eigenvalues of $A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ satisfy its characteristic equation, namely

$(\lambda-2)^2 (\lambda-4) = 0$. Verify the Cayley-Hamilton Theorem for this matrix and find the lowest degree factor of the characteristic polynomial that vanishes when A is substituted for λ (and constants are multiplied by the identity matrix I_3 as necessary). The characteristic polynomial still factorizes when A is substituted for λ (because all products involved commute) but now no linear factor vanishes even though a product of linear factors vanishes. Verify this and explain it in view of the fact that when λ is a field element at least one linear factor of the characteristic polynomial must vanish for the characteristic equation to be satisfied.

Solution

If $A := \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ then $A-2 = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $A-4 = \begin{bmatrix} -2 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Hence neither linear factor of the characteristic equation vanishes for $\lambda = A$.

There are two quadratic factors: $(A-2)^2 = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ and $(A-2)(A-4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence $(\lambda-2)(\lambda-4)$ is the lowest degree factor of the characteristic polynomial that vanishes for $\lambda = A$. Therefore $(\lambda-2)^2(\lambda-4)$ also vanishes for $\lambda = A$, so the Cayley-Hamilton Theorem is satisfied.

The reason why no linear factor vanishes is that it is possible for the product of two nonzero matrices to be zero (as illustrated here), which cannot happen in a field. (The set of all $n \times n$ matrices forms a ring with zero divisors, not a field.) The rule for fields that if a product vanishes then at least one factor must vanish does not apply to rings, and hence not to matrices.

Maple Solution